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ORIGINAL RESEARCH

A New Modified Liu Ridge-Type Estimator for the Linear Regression Model: Simulation and Application

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Abstract

Several efforts have been made to solve the multicollinearity problem, which arises from correlated regressors in the linear regression model. This is because the Ordinary Least Squares (OLS) Estimator becomes inefficient in the presence of multicollinearity. In this paper, a new modified Liu Ridge-Type estimator called Liu-Dawoud-Kibria is proposed in place of the OLS in estimating the parameters of the general linear model. The theoretical comparison and simulation study results show that the proposed estimator outperforms others under some conditions, using the mean squares error criterion. A real-life dataset is used to bolster the findings of the paper.

Keywords

Liu Dawoud-Kibria, Multicollinearity, OLS estimator, Ridge regression estimator, Monte carlo simulation, Mean square error

Introduction

One of the multiple linear regression model assumptions is that the explanatory variables are independent of each other. However, according to Frisch [1], this independence assumption is often violated in real-life situations leading to the Multicollinearity problem. The ordinary least squares (OLS) estimator has been regarded as one of the most important ways of estimating the parameters of the general linear model since it has minimum variance. But, in the presence of Multicollinearity, the OLSE is no longer a good estimator. Multicollinearity is generally agreed to be present if there is an approximately linear relationship (i.e., shared variance) among some predictor variables in the data [2]. The term multicollinearity refers to a situation where there is an exact (or nearly exact) linear relation among two or more explanatory variables [3]. One of the tests to ascertain the presence of multicollinearity in a dataset is the variance inflation factor (VIF). The VIF values ranging between one and five are moderate and represent a medium level of collinearity. Values of more than five are said to be highly collinear, while values above ten are considered to be extreme. Pairwise correlation of two independent variables can also be used to detect if multicollinearity is present in a dataset. An absolute correlation value of 0.7 and above indicates the presence of multicollinearity.

However, many methods have been proposed to solve this problem by various researchers (Hoerl and Kennard [4]; Liu [5]; Kibria [6]; Sakallioglu and Kaciranlar [7]; Baye and Parker [8]; Yang and Chang [9]; Wu and Yang [10]; Dorugade [11]; Lukman, et al. [12]); Lukman, et al. [13]; Kibria and Lukman [14]; Dawoud and Kibria [15]) and recently Owolabi, et al. [16,17]. This study proposes a new two-parameter estimator to circumvent the problem of Multicollinearity. Aside from multicollinearity, a few terms are defined in the table 1 below.

The organization of the paper is as follows. The model, the proposed and existing estimators are given in Section 2. A comparison of the proposed estimator with some existing ones is shown in section 3. The biasing parameter was obtained in Section 4, while a

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Table 1: Definition of some basic terms.

Regression analysis is a process used to estimate a function that predicts the value of the response variable in terms of values of other independent variables. The regression model often relies heavily on the underlying assumptions being satisfied. There are three types of regression, namely; - Simple linear regression studies the linear relationship between two variables, i.e., one dependent and one independent variable. - Multiple Linear regression studies the linear relationship between one dependent variable and more than one independent variables. - Nonlinear regression assumes that the relationship between a dependent variable and independent variables is not linear in regression parameters [18]
Ridge regression is one of the remedial measures for handling severe multicollinearity in the least-squares estimation. The ridge regression is a constrained version of least squares. It tackles the estimation problem by producing a biased estimator with small variances [18]. Ridge regression provides another alternative estimation method that may be used to advantage when the predictor variables are highly collinear [19]. The estimators produced are biased but tend to have a smaller mean squared error than OLS estimators [4].
Mean Squared Error Matrix measures the average of the error squares, i.e., the average squared difference between the estimated value and the actual values.
A **dispersion matrix**, also known as a covariance matrix, is a square matrix giving the covariance between each pair of elements of a given random vector.

Monte-Carlo simulation and a numeric example were conducted in Section 5. Section 6 contains concluding remarks.

The Models Specifications and New Estimator

The models and existing estimators

Consider the following linear regression model

$$y = X\beta + \varepsilon, \quad (1)$$

where y is a n × 1 vector of response, X is a n × p full column rank matrix, where *n* and *p* refer to the sample size and numbers of explanatory variables., β is a p × 1 vector of unknown parameters, ε is a n × 1 vector of errors supposed to be distributed with mean vector 0 and variance-covariance $\sigma^2 I$.

Based on the Gauss-Markov theorem, the ordinary least squares estimator (OLSE) is given as:

$$\hat{\beta} = (X'X)^{-1} X'y \quad (2)$$

The canonical form of Eq. (1) is rewritten as:

$$y = Z\alpha + \varepsilon \quad (3)$$

where $Z = XQ$, $\alpha = Q'\beta$ and Q is the orthogonal matrix whose columns constitute the eigenvectors of $X'X$. Then $Z'Z = Q'X'XQ = \Lambda = diag(\lambda_1,\ldots,\lambda_p)$,

where $\lambda_1 \geq \lambda_2 \geq \ldots\lambda_p > 0$ are the ordered eigenvalues $X'X$. The ordinary least square estimator (OLSE) of equation (2) can be defined as:

$$\hat{\alpha} = \Lambda^{-1}X'y \quad (4)$$

And the Mean Square Error matrix (MSEM) of ˆ is defined as

$$MSEM(\hat{\alpha}) = \Lambda^{-1}\sigma^2 \quad (5)$$

The Ordinary Ridge Regression Estimator by Hoerl and Kennard [4] is given as:

$$\hat{\alpha}_k = AX'y \quad (6)$$

where $A = (\Lambda + kI)^{-1}$ and k is a non-negative biasing parameter. And the MSEM is given as:

$$MSEM(\hat{\alpha}_k) = \sigma^2 A\Lambda A + (A\Lambda - I)\alpha\alpha'(A\Lambda - I)' \quad (7)$$

The Liu estimator is defined as:

$$\hat{\alpha}_d = E\hat{\alpha} \quad (8)$$

where $E = (\Lambda + I)^{-1}(\Lambda + dI)$ and d is a biasing parameter of Liu Estimator. The MSEM of $\hat{\alpha}_d$ is defined as:

$$MSEM(\hat{\alpha}_d) = \sigma^2 E\Lambda^{-1}E + (E - I)\alpha\alpha'(E - I)' \quad (9)$$

The Kibria-Lukman (KL) estimator is defined as:

$$\hat{\alpha}_{KL} = P\hat{\alpha} \quad (10)$$

where $P = (\Lambda + kI)^{-1}(\Lambda - kI)$ and the MSEM of $\hat{\alpha}_{KL}$ is given as

$$MSEM(\hat{\alpha}_{KL}) = \sigma^2 P\Lambda^{-1}P + (P - I)\alpha\alpha'(P - I)' \quad (11)$$

The Modified Ridge Type (MRT) estimator is defined as:

$$\hat{\alpha}_{MRT} = H\hat{\alpha} \quad (12)$$

where $H = \Lambda(\Lambda + k(1 + d)I)^{-1}$ and the MSEM is given as

$$MSEM(\hat{\alpha}_{MRT}) = \sigma^2 H\Lambda^{-1}H + (H - I)\alpha\alpha'(H - I)' \quad (13)$$

The Dawoud Kibria (DK) estimator is defined as:

$$\hat{\alpha}_{DK} = KM\hat{\alpha} \quad (14)$$

where $K = (\Lambda + k(1 + d))^{-1}$, $M = (\Lambda - k(1 + d))$ and the MSEM is given as

$$MSEM(\hat{\alpha}_{DK}) = \sigma^2 KM\Lambda^{-1}KM + (KM - I)\alpha\alpha'(KM - I)' \quad (15)$$

The proposed estimator

In this paper, we proposed a New Biasing Estimator called Liu Dawoud-Kibria Estimator $(\hat{\alpha}_{LDK})$ by following a method similar to that proposed by Liu [5]; Kaciranlar, et al. [20]; Yang and Chang [9], and Dawoud, et al. [21]. The proposed New Biasing Liu Dawoud-Kibria estimator for α is obtained by replacing $\hat{\alpha}_{DK}$ with $\hat{\alpha}$ in the Liu estimator, and it becomes as follows:

$$\hat{\alpha}_{LDK} = WS\hat{\alpha} \quad (16)$$

Where $W = (\Lambda + I)^{-1}(\Lambda + dI)$ $S = (\Lambda + k(1 + d)I)^{-1}(\Lambda - k(1 + d)I)$, d and k are the biasing parameters.

The properties of the new estimator, namely; the bias vector, covariance, and mean squared error matrix (MSEM) of the proposed estimator, are given as follows:

$$B(\hat{\alpha}_{LDK}) = (WS - I)\alpha \quad (17)$$

$$D(\hat{\alpha}_{LDK}) = \sigma^2 WS\Lambda^{-1}WS \quad (18)$$

$$MSEM(\hat{\alpha}_{LDK}) = \sigma^2 WS\Lambda^{-1}WS + (WS - I)\alpha\alpha'(WS - I) \quad (19)$$

The following lemmas will be used to make some theoretical comparisons among estimators in the next section.

Lemma 1: Let n x n matrices M > 0, N > 0 (or N > 0), then N > M if and only if $\lambda_1(NM^{-1}) < 1$ where $\lambda_1(NM^{-1}) < 1$ is the largest eigenvalue of matrix NM^{-1} [22].

Lemma 2: Let M be an n x n positive definite matrix, that is M > 0, and α be some vector, then $M - \alpha\alpha' \geq 0$ if and only if $\alpha' M^{-1}\alpha \leq 1$ [23].

Lemma 3: Let $\hat{\alpha}_i = A_i y$ i = 1, 2 be two linear estimators of α. Suppose that $D = \operatorname{cov}(\hat{\alpha}_1) - \operatorname{cov}(\alpha_2) > 0$, where $\operatorname{cov}(\hat{\alpha}_1)$ i = 1, 2 denotes the covariance matrix of ˆ and $b_i = Bias(\hat{\alpha}_i) = (A_i X - I)\alpha$, i = 1, 2. Consequently,

$$\Delta(\hat{\alpha}_1 - \hat{\alpha}_2) = MSEM(\hat{\alpha}_1) - MSEM(\hat{\alpha}_2) = \sigma^2 D + b_1 b_1' - b_2 b_2' > 0 \quad (20)$$

If and only if $b_2'\left[\sigma^2 D + b_1 b_1'\right]^{-1} b_2 < 1$, where $MSEM(\hat{\alpha}_i) = \operatorname{cov}(\hat{\alpha}_i) + b_i b_i'$ [24].

Comparison among the Estimators

Theoretical Comparisons among the Proposed LDK Estimator and the OLSE, Ordinary Ridge Regression (ORR), Liu Estimators, Kibria-Lukman (KL) Estimator, Dawoud-Kibria (DK), and Modified Ridge-Type (MRT) Estimator.

Theorem 1: The estimator $\hat{\alpha}_{LDK}$ is superior to the estimator $\hat{\alpha}$ if and only if

$$\alpha'(WS - I)'\left\lfloor \sigma^2\left(\Lambda^{-1} - WS\Lambda^{-1}WS\right)\right\rfloor \alpha(WS - I) < 1 \quad (21)$$

Proof: The difference between the dispersion matrices is given as

$$D(\hat{\alpha}) - D(\hat{\alpha}_{LDK}) = \sigma^2\left(\Lambda^{-1} - WS\Lambda^{-1}WS\right)$$

$$= \sigma^2 diag\left\{\frac{1}{\lambda_i} - \frac{(\lambda_i + d)^2(\lambda_i - k(1+d))^2}{\lambda_i(\lambda_i + 1)^2(\lambda_i + k(1+d))^2}\right\}_{i=1}^{p} \quad (22)$$

$\Lambda^{-1} - WS\Lambda^{-1}WS$ will be positively definite if and only if

$$(\lambda_i + 1)^2(\lambda_i + k(1+d))^2 - (\lambda_i + d)^2(\lambda_i - k(1+d))^2 > 0.$$

Theorem 2: The estimator $\hat{\alpha}_{LDK}$ is superior to the estimator $\hat{\alpha}_k$ if and only if

$$\alpha'(WS - I)'\left\lfloor \sigma^2\left(A\Lambda A - WS\Lambda^{-1}WS\right)\right\rfloor \alpha(WS - I) < 1 \quad (23)$$

Proof: The difference between the dispersion matrices is given as

$$D(\hat{\alpha}_k) - D(\hat{\alpha}_{LDK}) = \sigma^2\left(A\Lambda A - WS\Lambda^{-1}WS\right)$$

$$= \sigma^2 diag\left\{\frac{\lambda_i}{(\lambda_i + k)^2} - \frac{(\lambda_i + d)^2(\lambda_i - k(1+d))^2}{\lambda_i(\lambda_i + 1)^2(\lambda_i + k(1+d))^2}\right\}_{i=1}^{p} \quad (24)$$

AΛA - $WS\Lambda^{-1}WS$ will be positively definite if and only if

$$\lambda_i^2(\lambda_i + 1)^2(\lambda_i + k(1+d))^2 - (\lambda_i + d)^2(\lambda_i + k)^2(\lambda_i - k(1+d))^2 > 0.$$

Theorem 3: The estimator $\hat{\alpha}_{LDK}$ is superior to the estimator $\hat{\alpha}_d$ if and only if

$$\alpha'(WS - I)'\left\lfloor \sigma^2\left(E\Lambda^{-1}E - WS\Lambda^{-1}WS\right)\right\rfloor \alpha(WS - I) < 1 \quad (25)$$

Proof: The difference between the dispersion matrices is given as

$$D(\hat{\alpha}_d) - D(\hat{\alpha}_{LDK}) = \sigma^2\left(E\Lambda^{-1}E - WS\Lambda^{-1}WS\right)$$

$$= \sigma^2 diag\left\{\frac{(\lambda_i + d)^2}{\lambda_i(\lambda_i + 1)^2} - \frac{(\lambda_i + d)^2(\lambda_i - k(1+d))^2}{\lambda_i(\lambda_i + 1)^2(\lambda_i + k(1+d))^2}\right\}_{i=1}^{p} \quad (26)$$

$E\Lambda^{-1}E$ - $WS\Lambda^{-1}WS$ will be positively definite if and only if

$$(\lambda_i + d)^2(\lambda_i + k(1+d))^2 - (\lambda_i + d)^2(\lambda_i - k(1+d))^2 > 0.$$

Theorem 4: The estimator $\hat{\alpha}_{LDK}$ is superior to the estimator $\hat{\alpha}_{KL}$ if and only if

$$\alpha'(WS - I)'\left\lfloor \sigma^2\left(\mathrm{P}\Lambda^{-1}\mathrm{P} - WS\Lambda^{-1}WS\right)\right\rfloor \alpha(WS - I) < 1 \quad (27)$$

Proof: The difference between the dispersion matrices is given as

$$D(\hat{\alpha}_{KL}) - D(\hat{\alpha}_{LDK}) = \sigma^2\left(\mathrm{P}\Lambda^{-1}\mathrm{P} - WS\Lambda^{-1}WS\right)$$

$$= \sigma^2 diag\left\{\frac{(\lambda_i - k)^2}{\lambda_i(\lambda_i + k)^2} - \frac{(\lambda_i + d)^2(\lambda_i - k(1+d))^2}{\lambda_i(\lambda_i + 1)^2(\lambda_i + k(1+d))^2}\right\}_{i=1}^{p} \quad (28)$$

$\mathrm{P}\Lambda^{-1}\mathrm{P}$ - $WS\Lambda^{-1}WS$ will be positively definite if and only if

$$(\lambda_i - k)^2(\lambda_i + 1)^2(\lambda_i + k(1+d))^2 - (\lambda_i + d)^2(\lambda_i + k)^2(\lambda_i - k(1+d))^2 > 0.$$

Theorem 5: The estimator $\hat{\alpha}_{LDK}$ is superior to the estimator $\hat{\alpha}_{MRT}$ if and only if

$$\alpha'(WS - I)'\left\lfloor \sigma^2\left(\mathrm{H}\Lambda^{-1}\mathrm{H} - WS\Lambda^{-1}WS\right)\right\rfloor \alpha(WS - I) < 1 \quad (29)$$

Proof: The difference between the dispersion matrices is given as

$$D(\hat{\alpha}_{MRT}) - D(\hat{\alpha}_{LDK}) = \sigma^2\left(\mathrm{H}\Lambda^{-1}\mathrm{H} - WS\Lambda^{-1}WS\right)$$

$$= \sigma^2 diag\left\{\frac{\lambda_i}{(\lambda_i + k(1+d))^2} - \frac{(\lambda_i + d)^2(\lambda_i - k(1+d))^2}{\lambda_i(\lambda_i + 1)^2(\lambda_i + k(1+d))^2}\right\}_{i=1}^{p} \quad (30)$$

HΛ^{-1}H - *WS*Λ^{-1}*WS* will be positively definite if and only if $\lambda_i^2(\lambda_i+1)^2-(\lambda_i+d)^2(\lambda_i-k(1+d))^2>0.$

Theorem 6: The estimator $\hat{\alpha}_{LDK}$ is superior to the estimator $\hat{\alpha}_{DK}$ if and only if

$$\alpha'(WS-I)'\left[\sigma^2\left(\mathrm{P}\Lambda^{-1}\mathrm{P}-WS\Lambda^{-1}WS\right)\right]\alpha(WS-I)<1 \quad (31)$$

Proof: The difference between the dispersion matrices is given as

$$D(\hat{\alpha}_{DK})-D(\hat{\alpha}_{LDK})=\sigma^2\left(\mathrm{KM}\Lambda^{-1}\mathrm{KM}-WS\Lambda^{-1}WS\right)$$

$$(\lambda_i+1)^2(\lambda_i-k(1+d))^2-(\lambda_i+d)^2(\lambda_i-k(1+d))^2>0. \quad (32)$$

KMΛ^{-1}KM - *WS*Λ^{-1}*WS* will be positively definite if and only if

$$(\lambda_i+1)^2(\lambda_i-k(1+d))^2-(\lambda_i+d)^2(\lambda_i-k(1+d))^2>0.$$

Determination of the Parameters k and d

Various researchers have introduced different estimators of *k* and *d* for different kinds of regression models. Some of these authors are Hoerl and Kennard [4]; Dorugade [25]; Lukman and Ayinde [26]; Aslam and Ahmad [27], among others.The optimal values of *k* and *d* for the proposed estimator are obtainedthus. In determining the optimal value of *k*, *d* is fixed. The optimal value of the *k* can be considered to be that *k* that minimizes:

$$MSEM(\hat{\alpha}_{LDK})=\sigma^2 WS\Lambda^{-1}WS+(WS-I)\alpha\alpha'(WS-I)$$

$$g(k,d)=MSEM(\hat{\alpha}_{LDK})=tr\left[MSEM(\hat{\alpha}_{LDK})\right]$$

$$g(k,d)=\sigma^2\sum_i^p\frac{(\lambda_i+d)^2(\lambda_i-k(1+d))^2}{\lambda_i(\lambda_i+k(1+d))^2(\lambda_i+1)^2}+\sum_i^p\frac{\left[2k(\lambda_i+\lambda_i d+d)+k(d^2+1)+\lambda_i(1-d)\right]^2\alpha_i^2}{(\lambda_i+k(1+d))^2(\lambda_i+1)^2} \quad (33)$$

Taking the partial derivative of the function *g*(*k,d*) with respect to k gives

$$\frac{\partial g(k,d)}{\partial k}=2\sigma^2\sum_{i=1}^p\frac{(\lambda_i+d)^2(\lambda_i-k(d+1))(d+1)}{\lambda_i(\lambda_i+k(d+1))^2(\lambda_i+1)^2}-2\sigma^2\sum_{i=1}^p\frac{(\lambda_i+d)^2(\lambda_i-k(d+1))^2(d+1)}{\lambda_i(\lambda_i+k(d+1))^3(\lambda_i+1)^2}$$
$$+2\sum_{i=1}^p\frac{\left[2k(\lambda_i+\lambda_i d+d)-\lambda_i(d-1)+k(d^2+1)\right]\alpha_i^2}{(\lambda_i+k(d+1))^2(\lambda_i+1)^2}-2\sum_{i=1}^p\frac{\left[2k(\lambda_i+\lambda_i d+d)-\lambda_i(d-1)+k(d^2+1)\right]^2(d+1)\alpha_i^2}{(\lambda_i+k(d+1))^3(\lambda_i+1)^2} \quad (34)$$

Let $\frac{\partial g(k,d)}{\partial k}=0;$

$$k=\frac{\sigma^2\lambda_i(\lambda_i+d)-\alpha_i^2\lambda_i^2(1-d)}{\sigma^2 d(d+1)+\sigma^2\lambda_i(d+1)+\alpha_i^2\lambda_i(d^2+1)+2\alpha_i^2\lambda_i(\lambda_i d+d+\lambda_i)} \quad (35)$$

For practical purposes, σ² and α_i^2 are replaced with $\hat{\sigma}^2$ and $\hat{\alpha}_i^2$, respectively. Consequently, (30) becomes

$$\hat{k}=\frac{\hat{\sigma}^2\lambda_i(\lambda_i+d)-\hat{\alpha}_i^2\lambda_i^2(1-d)}{\hat{\sigma}^2 d(d+1)+\hat{\sigma}^2\lambda_i(d+1)+\hat{\alpha}_i^2\lambda_i(d^2+1)+2\hat{\alpha}_i^2\lambda_i(\lambda_i d+d+\lambda_i)} \quad (36)$$

and,

$$\hat{k}_{\min}=\min\left[\frac{\hat{\sigma}^2\lambda_i(\lambda_i+d)-\hat{\alpha}_i^2\lambda_i^2(1-d)}{\hat{\sigma}^2 d(d+1)+\hat{\sigma}^2\lambda_i(d+1)+\hat{\alpha}_i^2\lambda_i(d^2+1)+2\hat{\alpha}_i^2\lambda_i(\lambda_i d+d+\lambda_i)}\right]_{i=1}^p \quad (37)$$

and the biasing parameters *d* proposed by Liu [5] in equation (38) will be adopted in this study. It is given as follows:

$$d=\frac{\lambda_i(\alpha_i^2-\sigma^2)}{\sigma^2+\lambda_i\alpha_i^2} \quad (38)$$

For practical purposes, σ^2 and α_i^2 can be considered as $\hat{\sigma}^2$ and $\hat{\alpha}_i^2$, respectively. Consequently, (38) becomes

$$d=\frac{\lambda_i(\hat{\alpha}_i^2-\hat{\sigma}^2)}{\hat{\sigma}^2+\lambda_i\hat{\alpha}_i^2} \quad (39)$$

Simulation and Application

Simulation technique

The simulation procedure used by McDonald and Galarneau [28]; Wichern and Churchill [29]; Gibbons [30]; Lukman and Ayinde [26] was utilized to generate the predictor variables in this study: This is given as:

$$x_{ij} = \left(1-\rho^2\right)^{1/2} z_{ij} + \rho z_{i,p+1},\ i = 1,2,\ldots,n,\ j = 1,2,\ldots,p. \quad (40)$$

where z_{ij} is an independent standard normal distribution with mean zero and unit variance, ρ is the correlation between any two explanatory variables, and p is the number of explanatory variables. For this study, we considered the values of ρ to be 0.8, 0.9, 0.95, and 0.99. Also, explanatory variables (p) were taken to be three (3) and seven (7) for the simulation study. The error terms u_t were generated following Firinguetti [31] such that $u_t \sim N\left(0,\sigma^2 I\right)$. The values of $\beta'\beta = 1$ Newhouse and Oman [32]. The standard deviations in this simulation study were σ = 3, 5, and 10.

Simulation results from discussion

Table 2 and Table 3 show that as σ and ρ increase, the estimated MSE values increase. Likewise, as n increases, the estimated MSE values decrease. As expected and from several simulations and empirical research, when the multicollinearity problem exists, the OLS estimator gives the highest MSE values and performs the worst among all estimators. Additionally, the results show

Table 2: Estimated MSE when p = 3, n = 50.

k	d	sigma	rho	OLS	RIDGE	LIU	K-L	MRT	DK	LDK
0.3	0.2	1	0.8	0.1363	0.1311	0.1234	0.1261	0.1301	0.1241	**0.1125**
			0.9	0.2462	0.2288	0.2046	0.212	0.2255	0.2057	**0.1713**
			0.95	0.4682	0.4064	0.3333	0.3492	0.3956	0.3293	**0.2355**
			0.99	2.2417	1.2173	0.7223	0.5101	1.0998	0.3689	**0.1264**
		3	0.8	1.227	1.1802	1.1098	1.1344	1.1712	1.1168	**1.0107**
			0.9	2.2158	2.0587	1.8412	1.9076	2.0293	1.8514	**1.5407**
			0.95	4.2136	3.6579	2.9999	3.1428	3.5604	2.964	**2.1191**
			0.99	20.175	10.956	6.5009	4.5913	9.8985	3.3199	**1.1375**
		5	0.8	3.4082	3.2782	3.0825	3.1509	3.2532	3.1019	**2.8071**
			0.9	6.1551	5.7185	5.1142	5.2987	5.637	5.1426	**4.2792**
			0.95	11.704	10.161	8.333	8.73	9.8901	8.2333	**5.8864**
			0.99	56.042	30.434	18.058	12.754	27.496	9.2219	**3.1597**
		10	0.8	13.633	13.113	12.329	12.603	13.012	12.407	**11.227**
			0.9	24.62	22.874	20.457	21.194	22.548	20.57	**17.116**
			0.95	46.818	40.644	33.332	34.92	39.56	32.933	**23.546**
			0.99	224.17	121.73	72.233	51.015	109.98	36.888	**12.639**
0.3	0.5	1	0.8	0.1363	0.1311	0.1281	0.1261	0.1287	0.1213	**0.1141**
			0.9	0.2462	0.2288	0.2197	0.212	0.2207	0.1968	**0.1758**
			0.95	0.4682	0.4064	0.3811	0.3492	0.3802	0.3016	**0.2463**
			0.99	2.2417	1.2173	1.1925	0.5101	0.9535	0.2188	**0.1217**
		3	0.8	1.227	1.1802	1.153	1.1344	1.1578	1.0909	**1.0256**
			0.9	2.2158	2.0587	1.9774	1.9076	1.9865	1.7703	**1.5816**
			0.95	4.2136	3.6579	3.4302	3.1428	3.4215	2.7145	**2.2167**
			0.99	20.175	10.956	10.733	4.5913	8.5816	1.9692	**1.0948**
		5	0.8	3.4082	3.2782	3.2026	3.1509	3.2161	3.03	**2.8484**
			0.9	6.1551	5.7185	5.4928	5.2987	5.5181	4.9173	**4.3931**
			0.95	11.704	10.161	9.5284	8.73	9.5043	7.5403	**6.1574**
			0.99	56.042	30.434	29.813	12.754	23.838	5.4701	**3.041**
		10	0.8	13.633	13.113	12.81	12.603	12.864	12.119	**11.393**
			0.9	24.62	22.874	21.971	21.194	22.072	19.669	**17.572**
			0.95	46.818	40.644	38.114	34.92	38.017	30.161	**24.63**
			0.99	224.17	121.73	119.25	51.015	95.351	21.881	**12.164**

0.3	0.8	1	0.8	0.1363	0.1311	0.133	0.1261	0.1272	0.1185	**0.1157**
			0.9	0.2462	0.2288	0.2354	0.212	0.2162	0.1882	**0.18**
			0.95	0.4682	0.4064	0.4323	0.3492	0.3657	0.2762	**0.2554**
			0.99	2.2417	1.2173	1.7822	0.5101	0.8349	0.1225	**0.0996**
		3	0.8	1.227	1.1802	1.1971	1.1344	1.1448	1.0656	**1.0399**
			0.9	2.2158	2.0587	2.1188	1.9076	1.9451	1.6928	**1.6195**
			0.95	4.2136	3.6579	3.8903	3.1428	3.2909	2.4857	**2.2982**
			0.99	20.175	10.956	16.04	4.5913	7.5144	1.1027	**0.896**
		5	0.8	3.4082	3.2782	3.3251	3.1509	3.1798	2.9597	**2.8882**
			0.9	6.1551	5.7185	5.8855	5.2987	5.4031	4.702	**4.4984**
			0.95	11.704	10.161	10.806	8.73	9.1413	6.9046	**6.3839**
			0.99	56.042	30.434	44.554	12.754	20.873	3.063	**2.489**
		10	0.8	13.633	13.113	13.3	12.603	12.719	11.838	**11.552**
			0.9	24.62	22.874	23.542	21.194	21.612	18.808	**17.993**
			0.95	46.818	40.644	43.226	34.92	36.565	27.619	**25.536**
			0.99	224.17	121.73	178.22	51.015	83.494	12.252	**9.9562**
0.6	0.2	1	0.8	0.1363	0.1263	0.1234	0.1167	0.1244	0.1131	**0.1026**
			0.9	0.2462	0.2132	0.2046	0.1827	0.2074	0.1721	**0.1436**
			0.95	0.4682	0.3564	0.3333	0.2604	0.3391	0.2314	**0.1664**
			0.99	2.2417	0.7679	0.7223	0.0801	0.6563	0.0327	**0.0153**
		3	0.8	1.227	1.1362	1.1098	1.0491	1.1193	1.0169	**0.921**
			0.9	2.2158	1.9183	1.8412	1.6431	1.8663	1.548	**1.2903**
			0.95	4.2136	3.208	2.9999	2.3436	3.0516	2.0827	**1.4966**
			0.99	20.175	6.9114	6.5009	0.721	5.907	0.2939	**0.1371**
		5	0.8	3.4082	3.1559	3.0825	2.9139	3.109	2.8244	**2.5574**
			0.9	6.1551	5.3284	5.1142	4.5638	5.1839	4.2996	**3.5836**
			0.95	11.704	8.9111	8.333	6.5101	8.4768	5.7852	**4.1572**
			0.99	56.042	19.198	18.058	2.0029	16.408	0.8163	**0.3806**
		10	0.8	13.633	12.623	12.329	11.655	12.436	11.296	**10.228**
			0.9	24.62	21.313	20.457	18.255	20.735	17.198	**14.333**
			0.95	46.818	35.645	33.332	26.041	33.907	23.141	**16.629**
			0.99	224.17	76.794	72.233	8.0117	65.634	3.2652	**1.5225**
0.6	0.5	1	0.8	0.1363	0.1263	0.1281	0.1167	0.1217	0.1081	**0.1018**
			0.9	0.2462	0.2132	0.2197	0.1827	0.1992	0.1575	**0.141**
			0.95	0.4682	0.3564	0.3811	0.2604	0.3154	0.1937	**0.1589**
			0.99	2.2417	0.7679	1.1925	0.0801	0.5301	0.0227	**0.0144**
		3	0.8	1.227	1.1362	1.153	1.0491	1.0947	0.9706	**0.913**
			0.9	2.2158	1.9183	1.9774	1.6431	1.7923	1.4156	**1.2664**
			0.95	4.2136	3.208	3.4302	2.3436	2.8383	1.7424	**1.4286**
			0.99	20.175	6.9114	10.733	0.721	4.7705	0.2033	**0.1281**
		5	0.8	3.4082	3.1559	3.2026	2.9139	3.0407	2.6954	**2.5351**
			0.9	6.1551	5.3284	5.4928	4.5638	4.9784	3.9317	**3.517**
			0.95	11.704	8.9111	9.5284	6.5101	7.8841	4.84	**3.9681**
			0.99	56.042	19.198	29.813	2.0029	13.251	0.5647	**0.3558**
		10	0.8	13.633	12.623	12.81	11.655	12.162	10.78	**10.138**
			0.9	24.62	21.313	21.971	18.255	19.913	15.726	**14.066**
			0.95	46.818	35.645	38.114	26.041	31.537	19.36	**15.873**
			0.99	224.17	76.794	119.25	8.0117	53.006	2.2591	**1.4231**

0.6	0.8	1	0.8	0.1363	0.1263	0.133	0.1167	0.1191	0.1032	**0.1008**
			0.9	0.2462	0.2132	0.2354	0.1827	0.1915	0.1441	**0.138**
			0.95	0.4682	0.3564	0.4323	0.2604	0.2941	0.1617	**0.1499**
			0.99	2.2417	0.7679	1.7822	0.0801	0.4374	0.0543	**0.0437**
		3	0.8	1.227	1.1362	1.1971	1.0491	1.071	0.9264	**0.9043**
			0.9	2.2158	1.9183	2.1188	1.6431	1.7228	1.2944	**1.2392**
			0.95	4.2136	3.208	3.8903	2.3436	2.6472	1.4547	**1.3476**
			0.99	20.175	6.9114	16.04	0.721	3.9366	0.4881	**0.3923**
		5	0.8	3.4082	3.1559	3.3251	2.9139	2.9747	2.5725	**2.5109**
			0.9	6.1551	5.3284	5.8855	4.5638	4.7853	3.595	**3.4414**
			0.95	11.704	8.9111	10.806	6.5101	7.3532	4.0406	**3.7432**
			0.99	56.042	19.198	44.554	2.0029	10.935	1.3556	**1.0897**
		10	0.8	13.633	12.623	13.3	11.655	11.898	10.288	**10.042**
			0.9	24.62	21.313	23.542	18.255	19.141	14.379	**13.764**
			0.95	46.818	35.645	43.226	26.041	29.413	16.163	**14.973**
			0.99	224.17	76.794	178.22	8.0117	43.741	5.4227	**4.3589**
0.7	0.2	1	0.8	0.1363	0.1247	0.1234	0.1137	0.1226	0.1097	**0.0996**
			0.9	0.2462	0.2083	0.2046	0.1738	0.2019	0.1622	**0.1355**
			0.95	0.4682	0.3419	0.3333	0.2361	0.323	0.2056	**0.1481**
			0.99	2.2417	0.6731	0.7223	0.0378	0.5677	0.02	**0.0103**
		3	0.8	1.227	1.1221	1.1098	1.0222	1.1028	0.9858	**0.893**
			0.9	2.2158	1.8748	1.8412	1.5634	1.8164	1.4584	**1.2164**
			0.95	4.2136	3.0769	2.9999	2.1241	2.9068	1.8495	**1.3317**
			0.99	20.175	6.0583	6.5009	0.3396	5.1093	0.1793	**0.0912**
		5	0.8	3.4082	3.1167	3.0825	2.8391	3.0632	2.7377	**2.4795**
			0.9	6.1551	5.2076	5.1142	4.3425	5.0455	4.0507	**3.3781**
			0.95	11.704	8.5469	8.333	5.9004	8.0744	5.1375	**3.6991**
			0.99	56.042	16.829	18.058	0.9434	14.192	0.4981	**0.2532**
		10	0.8	13.633	12.467	12.329	11.355	12.252	10.949	**9.9158**
			0.9	24.62	20.83	20.457	17.369	20.182	16.202	**13.511**
			0.95	46.818	34.188	33.332	23.602	32.298	20.55	**14.796**
			0.99	224.17	67.315	72.233	3.7737	56.77	1.9927	**1.0128**
0.7	0.5	1	0.8	0.1363	0.1247	0.1281	0.1137	0.1195	0.104	**0.098**
			0.9	0.2462	0.2083	0.2197	0.1738	0.1927	0.1462	**0.131**
			0.95	0.4682	0.3419	0.3811	0.2361	0.2975	0.1667	**0.137**
			0.99	2.2417	0.6731	1.1925	0.0378	0.4511	0.0472	**0.0266**
		3	0.8	1.227	1.1221	1.153	1.0222	1.0749	0.9336	**0.8784**
			0.9	2.2158	1.8748	1.9774	1.5634	1.7341	1.3139	**1.176**
			0.95	4.2136	3.0769	3.4302	2.1241	2.6776	1.4993	**1.2313**
			0.99	20.175	6.0583	10.733	0.3396	4.0594	0.4235	**0.2377**
		5	0.8	3.1167	3.2026	2.8391	3.2573	3.2607	2.4388	**2.3475**
			0.9	6.1551	5.2076	5.4928	4.3425	4.8167	3.6491	**3.2657**
			0.95	11.704	8.5469	9.5284	5.9004	7.4378	4.1648	**3.4201**
			0.99	56.042	16.829	29.813	0.9434	11.276	1.1763	**0.6601**
		10	0.8	13.633	12.467	12.81	11.355	11.942	10.369	**9.7529**
			0.9	24.62	20.83	21.971	17.369	19.266	14.595	**13.061**
			0.95	46.818	34.188	38.114	23.602	29.751	16.659	**13.68**
			0.99	224.17	67.315	119.25	3.7737	45.104	4.7054	**2.6404**

0.7	0.8	1	0.8	0.1363	0.1247	0.133	0.1137	0.1166	0.0986	**0.0964**
			0.9	0.2462	0.2083	0.2354	0.1738	0.1843	0.1318	**0.1263**
			0.95	0.4682	0.3419	0.4323	0.2361	0.2751	0.1347	**0.125**
			0.99	2.2417	0.6731	1.7822	0.0378	0.3674	0.1077	**0.0857**
		3	0.8	1.227	1.1221	1.1971	1.0222	1.0481	0.8843	**0.8633**
			0.9	2.2158	1.8748	2.1188	1.5634	1.6574	1.1835	**1.1333**
			0.95	4.2136	3.0769	3.8903	2.1241	2.4752	1.2112	**1.123**
			0.99	20.175	6.0583	16.04	0.3396	3.3064	0.9676	**0.7701**
		5	0.8	3.4082	3.1167	3.3251	2.8391	2.911	2.4554	**2.3968**
			0.9	6.1551	5.2076	5.8855	4.3425	4.6037	3.2868	**3.1471**
			0.95	11.704	8.5469	10.806	5.9004	6.8757	3.3642	**3.1191**
			0.99	56.042	16.829	44.554	0.9434	9.1844	2.6876	**2.1388**
		10	0.8	13.633	12.467	13.3	11.355	11.643	9.8194	**9.5847**
			0.9	24.62	20.83	23.542	17.369	18.414	13.145	**12.586**
			0.95	46.818	34.188	43.226	23.602	27.503	13.457	**12.476**
			0.99	224.17	67.315	178.22	3.7737	36.738	10.751	**8.5552**
0.9	0.2	1	0.8	0.1363	0.1217	0.1234	0.1081	0.1191	0.1032	**0.0938**
			0.9	0.2462	0.1992	0.2046	0.1575	0.1915	0.1441	**0.1205**
			0.95	0.4682	0.3154	0.3333	0.1937	0.2941	0.1617	**0.1171**
			0.99	2.2417	0.5301	0.7223	0.0227	0.4374	0.0543	**0.0197**
		3	0.8	1.227	1.0947	1.1098	0.9706	1.071	0.9264	**0.8396**
			0.9	2.2158	1.7923	1.8412	1.4156	1.7228	1.2944	**1.081**
			0.95	4.2136	2.8383	2.9999	1.7424	2.6472	1.4547	**1.0522**
			0.99	20.175	4.7705	6.5009	0.2033	3.9366	0.4881	**0.1753**
		5	0.8	3.4082	3.0407	3.0825	2.6954	2.9747	2.5725	**2.3309**
			0.9	6.1551	4.9784	5.1142	3.9317	4.7853	3.595	**3.0019**
			0.95	11.704	7.8841	8.333	4.84	7.3532	4.0406	**2.9224**
			0.99	56.042	13.251	18.058	0.5647	10.935	1.3556	**0.4867**
		10	0.8	13.633	12.162	12.329	10.78	11.898	10.288	**9.3209**
			0.9	24.62	19.913	20.457	15.726	19.141	14.379	**12.005**
			0.95	46.818	31.537	33.332	19.36	29.413	16.163	**11.689**
			0.99	224.17	53.006	72.233	2.2591	43.741	5.4227	**1.9465**
	0.5	1	0.8	0.1363	0.1217	0.1281	0.1081	0.1153	0.0964	**0.091**
			0.9	0.2462	0.1992	0.2197	0.1575	0.1808	0.1261	**0.1131**
			0.95	0.4682	0.3154	0.3811	0.1937	0.2662	0.1228	**0.1014**
			0.99	2.2417	0.5301	1.1925	0.0227	0.3386	0.1388	**0.0741**
		3	0.8	1.227	1.0947	1.153	0.9706	1.0369	0.864	**0.8132**
			0.9	2.2158	1.7923	1.9774	1.4156	1.6262	1.1316	**1.0139**
			0.95	4.2136	2.8383	3.4302	1.7424	2.3957	1.1038	**0.9101**
			0.99	20.175	4.7705	10.733	0.2033	3.0474	1.2476	**0.6643**
		5	0.8	3.4082	3.0407	3.2026	2.6954	2.88	2.3989	**2.2574**
			0.9	6.1551	4.9784	5.4928	3.9317	4.5169	3.1425	**2.8152**
			0.95	11.704	7.8841	9.5284	4.84	6.6547	3.066	**2.5276**
			0.99	56.042	13.251	29.813	0.5647	8.4651	3.4652	**1.8448**
		10	0.8	13.633	12.162	12.81	10.78	11.519	9.5932	**9.0265**
			0.9	24.62	19.913	21.971	15.726	18.067	12.568	**11.258**
			0.95	46.818	31.537	38.114	19.36	26.619	12.264	**10.11**
			0.99	224.17	53.006	119.25	2.2591	33.861	13.861	**7.3791**

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0.9	0.8	1	0.8	0.1363	0.1217	0.133	0.1081	0.1118	0.0902	**0.0881**
			0.9	0.2462	0.1992	0.2354	0.1575	0.171	0.1104	**0.1058**
			0.95	0.4682	0.3154	0.4323	0.1937	0.2422	0.0926	**0.0861**
			0.99	2.2417	0.5301	1.7822	0.0227	0.2703	0.2401	**0.1905**
		3	0.8	1.227	1.0947	1.1971	0.9706	1.0046	0.806	**0.787**
			0.9	2.2158	1.7923	2.1188	1.4156	1.5378	0.9888	**0.9473**
			0.95	4.2136	2.8383	3.8903	1.7424	2.1794	0.8309	**0.772**
			0.99	20.175	4.7705	16.04	0.2033	2.4324	2.1589	**1.712**
		5	0.8	3.4082	3.0407	3.3251	2.6954	2.7899	2.2373	**2.1843**
			0.9	6.1551	4.9784	5.8855	3.9317	4.2714	2.7455	**2.6302**
			0.95	11.704	7.8841	10.806	4.84	6.0539	2.3076	**2.144**
			0.99	56.042	13.251	44.554	0.5647	6.7567	5.9966	**4.7552**
		10	0.8	13.633	12.162	13.3	10.78	11.159	8.946	**8.7336**
			0.9	24.62	19.913	23.542	15.726	17.085	10.979	**10.518**
			0.95	46.818	31.537	43.226	19.36	24.216	9.23	**8.5758**
			0.99	224.17	53.006	178.22	2.2591	27.027	23.986	**19.02**

Table 3: Estimated MSE when p = 3, n = 100.

k	d	sigma	rho	OLS	RIDGE	LIU	K-L	MRT	DK	LDK
0.3	0.2	1	0.8	0.0622	0.0611	0.0594	0.06	0.0609	0.0596	**0.057**
			0.9	0.1131	0.1093	0.1036	0.1056	0.1086	0.1042	**0.0955**
			0.95	0.2163	0.2023	0.1828	0.1889	0.1997	0.1839	**0.1556**
			0.99	1.0445	0.7667	0.5345	0.5329	0.7248	0.4649	**0.2416**
		3	0.8	0.5596	0.5499	0.5345	0.5403	0.548	0.5366	**0.5127**
			0.9	1.0178	0.9839	0.9322	0.9506	0.9774	0.9378	**0.8593**
			0.95	1.9464	1.8211	1.6447	1.7002	1.7976	1.6549	**1.4001**
			0.99	9.4002	6.9	4.8107	4.7956	6.5235	4.1841	**2.1739**
		5	0.8	1.5543	1.5275	1.4848	1.5009	1.5222	1.4905	**1.424**
			0.9	2.8273	2.7332	2.5896	2.6407	2.7149	2.6049	**2.3869**
			0.95	5.4066	5.0586	4.5687	4.7227	4.9932	4.597	**3.8892**
			0.99	26.112	19.167	13.363	13.321	18.121	11.623	**6.0386**
		10	0.8	6.2173	6.11	5.9393	6.0036	6.0889	5.9618	**5.6961**
			0.9	11.309	10.933	10.358	10.563	10.86	10.42	**9.5474**
			0.95	21.626	20.235	18.275	18.891	19.973	18.388	**15.557**
			0.99	104.45	76.667	53.452	53.285	72.483	46.49	**24.154**
0.3	0.5	1	0.8	0.0622	0.0611	0.0604	0.06	0.0606	0.059	**0.0574**
			0.9	0.1131	0.1093	0.1071	0.1056	0.1075	0.1021	**0.0967**
			0.95	0.2163	0.2023	0.195	0.1889	0.1959	0.1766	**0.1594**
			0.99	1.0445	0.7667	0.7055	0.5329	0.6683	0.3779	**0.258**
		3	0.8	0.5596	0.5499	0.5438	0.5403	0.5452	0.531	**0.5162**
			0.9	1.0178	0.9839	0.9639	0.9506	0.9677	0.9188	**0.8704**
			0.95	1.9464	1.8211	1.7548	1.7002	1.7631	1.5894	**1.4342**
			0.99	9.4002	6.9	6.3497	4.7956	6.015	3.4012	**2.322**
		5	0.8	1.5543	1.5275	1.5107	1.5009	1.5144	1.4749	**1.4337**
			0.9	2.8273	2.7332	2.6774	2.6407	2.6879	2.5523	**2.4177**
			0.95	5.4066	5.0586	4.8743	4.7227	4.8976	4.4148	**3.9838**
			0.99	26.112	19.167	17.638	13.321	16.708	9.4477	**6.4499**
		10	0.8	6.2173	6.11	6.0427	6.0036	6.0574	5.8998	**5.7348**
			0.9	11.309	10.933	10.71	10.563	10.752	10.209	**9.6709**
			0.95	21.626	20.235	19.497	18.891	19.59	17.659	**15.935**
			0.99	104.45	76.667	70.552	53.285	66.834	37.791	**25.799**

0.3	0.8	1	0.8	0.0622	0.0611	0.0615	0.06	0.0603	0.0584	**0.0578**
			0.9	0.1131	0.1093	0.1107	0.1056	0.1065	0.1	**0.0979**
			0.95	0.2163	0.2023	0.2076	0.1889	0.1922	0.1696	**0.1629**
			0.99	1.0445	0.7667	0.9008	0.5329	0.6183	0.306	**0.2651**
		3	0.8	0.5596	0.5499	0.5532	0.5403	0.5424	0.5255	**0.5196**
			0.9	1.0178	0.9839	0.9961	0.9506	0.9581	0.9003	**0.8812**
			0.95	1.9464	1.8211	1.8685	1.7002	1.7297	1.5264	**1.466**
			0.99	9.4002	6.9	8.1072	4.7956	5.5649	2.7536	**2.3861**
		5	0.8	1.5543	1.5275	1.5368	1.5009	1.5066	1.4596	**1.4432**
			0.9	2.8273	2.7332	2.7668	2.6407	2.6614	2.5007	**2.4476**
			0.95	5.4066	5.0586	5.1902	4.7227	4.8048	4.24	**4.072**
			0.99	26.112	19.167	22.52	13.321	15.458	7.649	**6.6279**
		10	0.8	6.2173	6.11	6.1472	6.0036	6.0262	5.8384	**5.7728**
			0.9	11.309	10.933	11.067	10.563	10.645	10.003	**9.7903**
			0.95	21.626	20.235	20.761	18.891	19.219	16.96	**16.288**
			0.99	104.45	76.667	90.08	53.285	61.832	30.596	**26.511**
0.6	0.2	1	0.8	0.0622	0.0601	0.0594	0.058	0.0597	0.0572	**0.0547**
			0.9	0.1131	0.1058	0.1036	0.0987	0.1044	0.0961	**0.0881**
			0.95	0.2163	0.1898	0.1828	0.1651	0.1851	0.1565	**0.1326**
			0.99	1.0445	0.5881	0.5345	0.2651	0.5341	0.1972	**0.105**
		3	0.8	0.5596	0.5405	0.5345	0.5218	0.5368	0.5146	**0.4918**
			0.9	1.0178	0.9518	0.9322	0.8881	0.9394	0.8643	**0.7924**
			0.95	1.9464	1.708	1.6447	1.4859	1.6658	1.408	**1.1928**
			0.99	9.4002	5.2929	4.8107	2.3853	4.8065	1.775	**0.944**
		5	0.8	1.5543	1.5014	1.4848	1.4495	1.4912	1.4294	**1.366**
			0.9	2.8273	2.6439	2.5896	2.4669	2.6094	2.4008	**2.2009**
			0.95	5.4066	4.7444	4.5687	4.1273	4.6272	3.911	**3.3131**
			0.99	26.112	14.702	13.363	6.6258	13.351	4.9303	**2.6219**
		10	0.8	6.2173	6.0056	5.9393	5.7978	5.9646	5.7176	**5.4637**
			0.9	11.309	10.575	10.358	9.8677	10.438	9.6032	**8.8032**
			0.95	21.626	18.978	18.275	16.509	18.509	15.644	**13.252**
			0.99	104.45	58.81	53.452	26.503	53.405	19.721	**10.487**
0.6	0.5	1	0.8	0.0622	0.0601	0.0604	0.058	0.0591	0.0561	**0.0545**
			0.9	0.1131	0.1058	0.1071	0.0987	0.1024	0.0923	**0.0875**
			0.95	0.2163	0.1898	0.195	0.1651	0.1784	0.1444	**0.1304**
			0.99	1.0445	0.5881	0.7055	0.2651	0.4662	0.1233	**0.0859**
		3	0.8	0.5596	0.5405	0.5438	0.5218	0.5314	0.504	**0.49**
			0.9	1.0178	0.9518	0.9639	0.8881	0.9213	0.8299	**0.7864**
			0.95	1.9464	1.708	1.7548	1.4859	1.6055	1.2988	**1.1732**
			0.99	9.4002	5.2929	6.3497	2.3853	4.1956	1.1091	**0.7724**
		5	0.8	1.5543	1.5014	1.5107	1.4495	1.476	1.3999	**1.361**
			0.9	2.8273	2.6439	2.6774	2.4669	2.5591	2.3051	**2.1844**
			0.95	5.4066	4.7444	4.8743	4.1273	4.4597	3.6077	**3.2587**
			0.99	26.112	14.702	17.638	6.6258	11.654	3.0808	**2.1454**
		10	0.8	6.2173	6.0056	6.0427	5.7978	5.9041	5.5996	**5.4437**
			0.9	11.309	10.575	10.71	9.8677	10.236	9.2202	**8.7372**
			0.95	21.626	18.978	19.497	16.509	17.839	14.431	**13.034**
			0.99	104.45	58.81	70.552	26.503	46.617	12.323	**8.5812**

0.6	0.8	1	0.8	0.0622	0.0601	0.0615	0.058	0.0585	0.0549	**0.0543**
			0.9	0.1131	0.1058	0.1107	0.0987	0.1004	0.0886	**0.0868**
			0.95	0.2163	0.1898	0.2076	0.1651	0.1721	0.1332	**0.128**
			0.99	1.0445	0.5881	0.9008	0.2651	0.4107	0.0739	**0.0647**
		3	0.8	0.5596	0.5405	0.5532	0.5218	0.526	0.4936	**0.4881**
			0.9	1.0178	0.9518	0.9961	0.8881	0.9037	0.7968	**0.7801**
			0.95	1.9464	1.708	1.8685	1.4859	1.5485	1.1981	**1.1512**
			0.99	9.4002	5.2929	8.1072	2.3853	3.6959	0.6642	**0.582**
		5	0.8	1.5543	1.5014	1.5368	1.4495	1.4611	1.3711	**1.3558**
			0.9	2.8273	2.6439	2.7668	2.4669	2.5103	2.2133	**2.1667**
			0.95	5.4066	4.7444	5.1902	4.1273	4.3015	3.3278	**3.1975**
			0.99	26.112	14.702	22.52	6.6258	10.266	1.845	**1.6166**
		10	0.8	6.2173	6.0056	6.1472	5.7978	5.8445	5.4841	**5.4229**
			0.9	11.309	10.575	11.067	9.8677	10.041	8.853	**8.6664**
			0.95	21.626	18.978	20.761	16.509	17.206	13.311	**12.79**
			0.99	104.45	58.81	90.08	26.503	41.065	7.3796	**6.4659**
0.7	0.2	1	0.8	0.0622	0.0597	0.0594	0.0573	0.0593	0.0564	**0.054**
			0.9	0.1131	0.1046	0.1036	0.0965	0.103	0.0935	**0.0858**
			0.95	0.2163	0.1859	0.1828	0.1579	0.1806	0.1483	**0.1257**
			0.99	1.0445	0.5425	0.5345	0.2074	0.4873	0.1448	**0.078**
		3	0.8	0.5596	0.5374	0.5345	0.5158	0.5332	0.5075	**0.4851**
			0.9	1.0178	0.9415	0.9322	0.8682	0.9273	0.8412	**0.7713**
			0.95	1.9464	1.6727	1.6447	1.4207	1.6252	1.3342	**1.1308**
			0.99	9.4002	4.8827	4.8107	1.8662	4.3853	1.3027	**0.7009**
		5	0.8	1.5543	1.4929	1.4848	1.4327	1.481	1.4097	**1.3472**
			0.9	2.8273	2.6151	2.5896	2.4117	2.5757	2.3365	**2.1423**
			0.95	5.4066	4.6464	4.5687	3.9463	4.5145	3.7061	**3.1409**
			0.99	26.112	13.563	13.363	5.1838	12.181	3.6186	**1.9468**
		10	0.8	6.2173	5.9714	5.9393	5.7309	5.9241	5.6386	**5.3885**
			0.9	11.309	10.461	10.358	9.6467	10.303	9.346	**8.5688**
			0.95	21.626	18.586	18.275	15.785	18.058	14.824	**12.563**
			0.99	104.45	54.252	53.452	20.735	48.726	14.474	**7.7867**
0.7	0.5	1	0.8	0.0622	0.0597	0.0604	0.0573	0.0586	0.0551	**0.0536**
			0.9	0.1131	0.1046	0.1071	0.0965	0.1008	0.0892	**0.0846**
			0.95	0.2163	0.1859	0.195	0.1579	0.1731	0.135	**0.122**
			0.99	1.0445	0.5425	0.7055	0.2074	0.4192	0.0807	**0.0569**
		3	0.8	0.5596	0.5374	0.5438	0.5158	0.5269	0.4954	**0.4816**
			0.9	1.0178	0.9415	0.9639	0.8682	0.9066	0.8022	**0.7604**
			0.95	1.9464	1.6727	1.7548	1.4207	1.5578	1.2143	**1.0972**
			0.99	9.4002	4.8827	6.3497	1.8662	3.7726	0.726	**0.5117**
		5	0.8	1.5543	1.4929	1.5107	1.4327	1.4636	1.3758	**1.3376**
			0.9	2.8273	2.6151	2.6774	2.4117	2.5184	2.2283	**2.1119**
			0.95	5.4066	4.6464	4.8743	3.9463	4.3272	3.3729	**3.0477**
			0.99	26.112	13.563	17.638	5.1838	10.48	2.0165	**1.4211**
		10	0.8	6.2173	5.9714	6.0427	5.7309	5.8543	5.5032	**5.3502**
			0.9	11.309	10.461	10.71	9.6467	10.073	8.9131	**8.4472**
			0.95	21.626	18.586	19.497	15.785	17.309	13.491	**12.19**
			0.99	104.45	54.252	70.552	20.735	41.918	8.0656	**5.6838**

0.7	0.8	1	0.8	0.0622	0.0597	0.0615	0.0573	0.0579	0.0538	**0.0533**
			0.9	0.1131	0.1046	0.1107	0.0965	0.0985	0.0851	**0.0834**
			0.95	0.2163	0.1859	0.2076	0.1579	0.1661	0.1229	**0.1181**
			0.99	1.0445	0.5425	0.9008	0.2074	0.3647	0.0421	**0.0372**
		3	0.8	0.5596	0.5374	0.5532	0.5158	0.5207	0.4835	**0.4781**
			0.9	1.0178	0.9415	0.9961	0.8682	0.8867	0.7652	**0.7491**
			0.95	1.9464	1.6727	1.8685	1.4207	1.4947	1.105	**1.062**
			0.99	9.4002	4.8827	8.1072	1.8662	3.2818	0.3778	**0.3338**
		5	0.8	1.5543	1.4929	1.5368	1.4327	1.4465	1.3429	**1.3279**
			0.9	2.8273	2.6151	2.7668	2.4117	2.463	2.1253	**2.0806**
			0.95	5.4066	4.6464	5.1902	3.9463	4.1518	3.0694	**2.9497**
			0.99	26.112	13.563	22.52	5.1838	9.1161	1.0491	**0.9269**
		10	0.8	6.2173	5.9714	6.1472	5.7309	5.7859	5.3712	**5.3114**
			0.9	11.309	10.461	11.067	9.6467	9.8519	8.5008	**8.3221**
			0.95	21.626	18.586	20.761	15.785	16.607	12.277	**11.798**
			0.99	104.45	54.252	90.08	20.735	36.464	4.1961	**3.7071**
0.9	0.2	1	0.8	0.0622	0.0591	0.0594	0.0561	0.0585	0.0549	**0.0526**
			0.9	0.1131	0.1024	0.1036	0.0923	0.1004	0.0886	**0.0813**
			0.95	0.2163	0.1784	0.1828	0.1444	0.1721	0.1332	**0.1131**
			0.99	1.0445	0.4662	0.5345	0.1233	0.4107	0.0739	**0.0412**
		3	0.8	0.5596	0.5314	0.5345	0.504	0.526	0.4936	**0.4719**
			0.9	1.0178	0.9213	0.9322	0.8299	0.9037	0.7968	**0.7309**
			0.95	1.9464	1.6055	1.6447	1.2988	1.5485	1.1981	**1.0164**
			0.99	9.4002	4.1956	4.8107	1.1091	3.6959	0.6642	**0.37**
		5	0.8	1.5543	1.476	1.4848	1.3999	1.4611	1.3711	**1.3105**
			0.9	2.8273	2.5591	2.5896	2.3051	2.5103	2.2133	**2.0299**
			0.95	5.4066	4.4597	4.5687	3.6077	4.3015	3.3278	**2.8229**
			0.99	26.112	11.654	13.363	3.0808	10.266	1.845	**1.0276**
		10	0.8	6.2173	5.9041	5.9393	5.5996	5.8445	5.4841	**5.2415**
			0.9	11.309	10.236	10.358	9.2202	10.041	8.853	**8.1192**
			0.95	21.626	17.839	18.275	14.431	17.206	13.311	**11.291**
			0.99	104.45	46.617	53.452	12.323	41.065	7.3796	**4.1099**
	0.5	1	0.8	0.0622	0.0591	0.0604	0.0561	0.0576	0.0533	**0.0519**
			0.9	0.1131	0.1024	0.1071	0.0923	0.0976	0.0834	**0.0791**
			0.95	0.2163	0.1784	0.195	0.1444	0.1632	0.118	**0.1068**
			0.99	1.0445	0.4662	0.7055	0.1233	0.3446	0.0312	**0.023**
		3	0.8	0.5596	0.5314	0.5438	0.504	0.5181	0.4785	**0.4653**
			0.9	1.0178	0.9213	0.9639	0.8299	0.8784	0.7498	**0.7109**
			0.95	1.9464	1.6055	1.7548	1.2988	1.4688	1.0612	**0.9597**
			0.99	9.4002	4.1956	6.3497	1.1091	3.1008	0.2798	**0.2059**
		5	0.8	1.5543	1.476	1.5107	1.3999	1.4392	1.329	**1.2922**
			0.9	2.8273	2.5591	2.6774	2.3051	2.4399	2.0826	**1.9743**
			0.95	5.4066	4.4597	4.8743	3.6077	4.08	2.9476	**2.6654**
			0.99	26.112	11.654	17.638	3.0808	8.6133	0.777	**0.5716**
		10	0.8	6.2173	5.9041	6.0427	5.5996	5.7569	5.3157	**5.1685**
			0.9	11.309	10.236	10.71	9.2202	9.7593	8.3302	**7.8967**
			0.95	21.626	17.839	19.497	14.431	16.32	11.79	**10.661**
			0.99	104.45	46.617	70.552	12.323	34.453	3.1074	**2.2859**

0.9	0.8	1	0.8	0.0622	0.0591	0.0615	0.0561	0.0568	0.0518	**0.0512**
			0.9	0.1131	0.1024	0.1107	0.0923	0.0949	0.0786	**0.077**
			0.95	0.2163	0.1784	0.2076	0.1444	0.1551	0.1046	**0.1005**
			0.99	1.0445	0.4662	0.9008	0.1233	0.2934	0.0133	**0.0122**
		3	0.8	0.5596	0.5314	0.5532	0.504	0.5105	0.4639	**0.4588**
			0.9	1.0178	0.9213	0.9961	0.8299	0.8541	0.7057	**0.691**
			0.95	1.9464	1.6055	1.8685	1.2988	1.3954	0.9397	**0.9034**
			0.99	9.4002	4.1956	8.1072	1.1091	2.6407	0.1189	**0.1082**
		5	0.8	1.5543	1.476	1.5368	1.3999	1.4179	1.2883	**1.2741**
			0.9	2.8273	2.5591	2.7668	2.3051	2.3725	1.9599	**1.9189**
			0.95	5.4066	4.4597	5.1902	3.6077	3.8759	2.6099	**2.5091**
			0.99	26.112	11.654	22.52	3.0808	7.3351	0.3301	**0.3001**
		10	0.8	6.2173	5.9041	6.1472	5.5996	5.6715	5.1528	**5.0956**
			0.9	11.309	10.236	11.067	9.2202	9.4898	7.839	**7.6751**
			0.95	21.626	17.839	20.761	14.431	15.503	10.439	**10.036**
			0.99	104.45	46.617	90.08	12.323	29.34	1.3197	**1.1998**

Note: For each row, the smallest MSE value is bolded.

Table 4: The results of regression coefficients and the corresponding MSE values.

	OLS	RIDGE	LIU	KL	MRT	DK	LDK
$\hat{\alpha}_0$	62.40537	8.5871	27.665	27.627	6.2282	27.588	-1.1164
$\hat{\alpha}_1$	1.551103	2.1046	1.9008	1.9088	2.1288	1.9092	-8.4346
$\hat{\alpha}_2$	0.510168	1.0648	0.8699	0.8685	1.08917	0.8689	-53.0519
$\hat{\alpha}_3$	0.101909	0.6680	0.4619	0.4678	0.69287	0.4682	-11.9168
$\hat{\alpha}_4$	-0.14406	0.3995	0.2080	0.2072	0.42342	0.2076	-29.14064
k		0.007676		0.000471	0.007676	0.000471	0.0000010298
d	-		0.442224		0.442224	0.001536	0.442224
MSE	4912.09	2989.820	2170.967	2170.960	2237	2170.960	**2170.959**

that the proposed LDK estimator performs best and dominates the rest of the estimators in this study. Thus, the findings agree with the theoretical results.

Numerical example

In this section, Portland cement data was used to demonstrate the performance of the proposed estimator. The Portland cement data was originally adopted by Woods, et al. [33] and was later adopted by Li and Yang [34] and Ayinde, et al. [35]. The data set is widely known as the Portland cement dataset. The regression model for these data is defined as:

$$y_i = \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \beta_4 X_4 + \varepsilon_i \quad (41)$$

where y_i = heat evolved after 180 days of curing measured in calories per gram of cement, X_1 = tricalcium aluminate, X_2 = tricalcium silicate, X_3 = tetracalcium aluminoferrite, and X_4 = β-dicalcium silicate. The variance inflation factors (VIFs) are 38.50, 254.42, 46.87, and 282.51, respectively. Eigenvalues of $X'X$ matrix are λ_1 = 44676.206, λ_2 = 5965.422, λ_3 = 809.952, and λ_4 = 105.419, and the condition number of $X\ X$ is approximately 424. The VIFs, the eigenvalues, and the condition number indicate severe Multicollinearity. The estimated parameters and the MSE values of the estimators are presented in Table 4.

From Table 4, the proposed estimator (LDK) performs best among other estimators as it gives the smallest MSE value. Just as observed in the simulation study results, the OLS estimator did not perform well in the presence of Multicollinearity as it has the highest MSE.

Concluding Remarks

In this paper, a new two-parameter estimator (LKD) is proposed. Theoretical comparison of the proposed with six other existing estimators shows the superiority of the proposed estimator. Results from the simulation study reveal that the proposed estimator performs better than other existing estimators used in this study under certain conditions, which further strengthens the theoretical study. Application to real-life dataset also reveals the dominance of the proposed estimator (LKD). The newly proposed estimator (LKD) is recommended for parameter estimation in the linear regression model in the presence of Multicollinearity.

Declaration of Competing Interest

The authors declare that they have no known competing interests.

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