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Target Frequency Analysis of functional MRI Data

Michael A. Frölich[1]*, Paul Jung[2] and Shannon Starr[2]

[1]Professor, Department of Anesthesiology, University of Alabama at Birmingham, USA

[2]Associate Professor, Department of Mathematics, University of Alabama at Birmingham, USA

***Corresponding author:** *Michael A. Frolich, Department of Anesthesiology, University of Alabama at Birmingham, 619 South 19th Street, Birmingham, AL 35249-6810, USA, Tel: (205) 975-0145, Fax: (205) 975-0145, E-mail: froelich@uab.edu*

Appendix – Mathematical Proof of Amplitude Distribution Theorem

Let x_s and x_t be *iid* $N(0,1)$ random variables for $s = 0\ldots, N-1$ and $t = 0\ldots, N-1$

In particular we have:

$$\text{Cov}\,(x_s, x_t) = \mathbb{E}[x_s, x_t] = \delta_{s,t} \tag{1}$$

Where

$$\delta_{s,t} = \begin{cases} 1 & \text{if s equals t, and} \\ 0 & \text{otherwise.} \end{cases}$$

We also have

$$\text{Var}\,(x_s) = \mathbb{E}\left[x_s^2\right] \text{ and } \text{Var}\,(x_s) = \mathbb{E}\left[x_s^2\right] \tag{2}$$

And, given a constant *c*,

$$c \cdot x_s \sim N(0,c) \text{ and } c \cdot x_t \sim N(0,c) \tag{3}$$

Under the linearity of expectations, we proceed as follows:

$$\mathbb{E}[Z_k, Z_l] = \mathbb{E}\left[\sum_{t=0}^{N-1} x_t e^{-i\left(\frac{2\pi}{N}\right)tk} \sum_{s=0}^{N-1} x_s e^{-i\left(\frac{2\pi}{N}\right)sl}\right]$$

$$= \sum_{t=0}^{N-1}\sum_{s=0}^{N-1} \mathbb{E}\left[x_t x_s e^{-i(2\pi/N)tk} e^{-i(2\pi/N)sl}\right]$$

$$= \sum_{t=0}^{N-1}\sum_{s=0}^{N-1} \mathbb{E}[x_t x_s] e^{-i(2\pi/N)tk} e^{-i(2\pi/N)sl}$$

$$= \sum_{t=0}^{N-1}\sum_{s=0}^{N-1} \delta_{t,s}\, e^{-i(2\pi/N)tk} e^{-i(2\pi/N)sl}$$

where we used Eq. (1) in the last step. One way to interpret the Kronecker delta is that, for fixed *t* if we sum on *s* over $0,\ldots,N-1$ the only

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value of s where the summand is not 0 is $s=t$. Therefore, we set $s = t$ in the summand, and remove the s - summation. This gives

$$\sum_{t=0}^{N-1}\sum_{s=0}^{N-1}\delta_{t,s}\, e^{-i(2\pi/N)tk} e^{-i(2\pi/N)sl} = \sum_{t=0}^{N-1} e^{-i(2\pi/N)tk} e^{-i(2\pi/N)tl}$$

By taking out the power t and combining exponentials, we have $e^{-i(2\pi/N)tk} e^{-i(2\pi/N)tl} = \left(e^{-i2\pi(k+l)/N}\right)^t$. Using this in out formula, we get

$$\mathbb{E}\left[Z_k, Z_l\right] = \sum_{t=0}^{N-1} e^{-i(2\pi/N)tk} e^{-i(2\pi/N)tl} = \sum_{t=0}^{N-1}\left(e^{-i2\pi(k+l)/N}\right)^t$$

Now, if $k+l$ satisfy $e^{(-i2\pi(k+l)/N)} = 1$ then,

$$\mathbb{E}\left[Z_k, Z_l\right] = \sum_{t=0}^{N-1}\left(e^{-i2\pi(k+l)/N}\right)^t = \sum_{t=0}^{N-1} 1^t = \mathrm{N} \tag{4}$$

On the other hand if $k+l$ is not a multiple of N, we use the Geometric sum formula:

$$1 + x + \cdots + x^{N-1} = \sum_{t=0}^{N-1} x^t = \frac{1-x^N}{1-x}$$

Using this with $x = e^{-i2\pi(k+l)/N}$, we have

$$\mathbb{E}\left[Z_k, Z_l\right] = \sum_{t=0}^{N-1}\left(e^{-i2\pi(k+l)/N}\right)^t = \frac{1-\left(e^{-i2\pi(k+l)/N}\right)^N}{1-e^{-i2\pi(k+l)/N}} = 0$$

In the last step we used the fact that for any integer, a multiple of $e^{-i2\pi\cdot a} = 1$. We obtained two different cases:

$$\mathbb{E}\left[Z_k, Z_l\right] = \begin{cases} N & \text{if } e^{-i2\pi(k+l)/N} = 1, \text{ i.e. } l = N-k \\ 0 & \text{otherwise} \end{cases}$$

As shorthand we may use the Kronecker delta function again to get:

$$\mathbb{E}\left[Z_k, Z_l\right] = N \cdot \delta_{l,N-k} \tag{5}$$

Notice the original formula takes the form

$$Z_k = \sum_{t=0}^{N-1} x_t e^{-it(2\pi/N)k} = \sum_{t=0}^{N-1} x_t \cos\left(\frac{2\pi kt}{N}\right) - i\sum_{t=0}^{N-1} x_t \sin\left(\frac{2\pi kt}{N}\right) = X_k - iY_k$$

We also have the similar formula for $l : Z_l = X_l - iY_l$. Therefore, what we have proved is a formula for

$$\mathbb{E}\left[Z_k, Z_l\right] = \mathbb{E}\left[\left(X_k - iY_k\right)\left(X_l - iY_l\right)\right] = \mathbb{E}\left[\mathrm{X_k X_l}\right] - \mathrm{i}\mathbb{E}\left[\mathrm{X_k Y_l}\right] - \mathrm{i}\mathbb{E}\left[\mathrm{Y_k X_l}\right] - \mathbb{E}\left[\mathrm{Y_k Y_l}\right]$$

Combining this with (5) gives

$$\mathbb{E}\left[X_k X_l\right] - i\mathbb{E}\left[X_k Y_l\right] - i\mathbb{E}\left[Y_k X_l\right] - \mathbb{E}\left[Y_k Y_l\right] = N \cdot \delta_{l,N-k} \tag{6}$$

Using the fact that cosine is an even function and sine is an odd function we have

$$X_{N-k} = X_k \text{ and } Y_{N-k} = -Y_k \tag{7}$$

Therefore, in (6), replacing k by $N-k$, we obtain

$$\mathbb{E}\left[X_k X_l\right] - i\mathbb{E}\left[X_k Y_l\right] + i\mathbb{E}\left[Y_k X_l\right] + \mathbb{E}\left[Y_k Y_l\right] = N \cdot \delta_{l,k} \tag{8}$$

Note that in the Kronecker delta function we replaced $N-k$ by k because $N-\left(N-k\right) = k$. similarly, changing (6) by replacing l with $N-l$ in the expectations and considering (7), we get

$$\mathbb{E}\left[X_k X_l\right] + i\mathbb{E}\left[X_k Y_l\right] - i\mathbb{E}\left[Y_k X_l\right] + \mathbb{E}\left[Y_k Y_l\right] = N \cdot \delta_{l,k} \tag{9}$$

Finally, replacing k by $N-k$ and l by $N-l$ in the expectations of (6), we obtain

$$\mathbb{E}\left[X_k X_l\right]+i\mathbb{E}\left[X_k Y_l\right]+i\mathbb{E}\left[Y_k X_l\right]-\mathbb{E}\left[Y_k Y_l\right]=N\cdot\delta_{l,N-k} \tag{10}$$

Adding equations (6) and (10) and dividing by 4, we get

$$\mathbb{E}\left[X_k X_l\right]=\frac{1}{2}N\left(\delta_{l,k}+\delta_{l,N-k}\right) \tag{11}$$

Adding just equations (6) and (9), we obtain

$$2\mathbb{E}\left[X_k X_l\right]+2i\mathbb{E}\left[Y_k X_l\right]=N\left(\delta_{l,k}+\delta_{l,N-k}\right)$$

But then using (11), we can see that this implies

$$\mathbb{E}\left[Y_k X_l\right]=0 \tag{12}$$

This equation is true for all k and l. It is also true if we switch the role of k and l. Therefore,

$$\mathbb{E}\left[X_k Y_l\right]=0 \tag{13}$$

Finally, substituting (11), (12) and (13) into (6), we get

$$\frac{1}{2}N\left(\delta_{l,k}+\delta_{l,N-k}\right)-0\cdot i-0\cdot i-\mathbb{E}\left[Y_k Y_l\right]=N\cdot\delta_{l,N-k}$$

Solving this equation, we obtain

$$\mathbb{E}\left[Y_k Y_l\right]=\frac{1}{2}N\left(\delta_{l,k}+\delta_{l,N-k}\right) \tag{14}$$

Now note that $\left(X_0,X_1,\ldots,X_{N-1},Y_0,Y_1,\ldots,Y_{N-1}\right)$ are all jointly Normal random variables, and all have mean zero. Therefore everything about their joint probability density function may be deduced from the variances and covariances (between the marginal random variables). In particular, for jointly Normal random variables, having all zero covariances means that the random variables are independent.

In (13), we see that

Cov $\left(X_k,Y_l\right)=\mathbb{E}\left[X_k,Y_l\right]-\mathbb{E}\left[X_k\right]\mathbb{E}\left[Y_l\right]=0$,

Thus any X_k and Y_l are independent, for each choice of k and l (including $k=l$). Also, note that for Normal random variables, pairwise independence implies full independence for a family of three or more variables. Using (11) and (14) to calculate variances and covariances between pairs of $X's$ and pairs of $Y's$, and recalling equality (2) and (3), we deduce that the family $(X_1,Y_1,X_2,Y_2,\ldots,X_{N/2-1},Y_{N/2-1})$ are all iid, $\mathcal{N}(0,N)$ because in (11), setting $k=l=0$ we obtain $\mathbb{E}[X_0\cdot X_0]=\frac{1}{2}N\left(\delta_{0,0}+\delta_{0,N}\right)$. N is interpreted as 0 in this context because x_t only goes up to $t=0,\ldots,N-1$ and using modular arithmetic $0\cong N$. Also $\mathrm{Var}\left(Y_0\right)=0$, as one can see by using (14). That is why we start at $k=1$ in (15).

Here $[N/2]$ equals the smallest integer n satisfying $n\geq N/2$. Therefore, we can see that

$$[N/2]-1=\begin{cases}(N/2)-1 & \text{if } N \text{ is even,}\\ (N-1)/2 & \text{if } N \text{ is odd.}\end{cases}$$

It is the largest integer m such that $m<N/2$. The reason for making this restriction is that if $1\leq k\leq m$ and $1\leq l\leq m$ then $k+l\leq 2m\leq N$, which means that $\delta_{l,N-k}=0$ for each pair from this set. So then we obtain

$$\mathbb{E}\left[X_k X_l\right]=\frac{1}{2}N\cdot\delta_{k,l}=\mathbb{E}\left[Y_k Y_l\right]$$

We note that the importance of stopping at $m<N/2$, and taking $k>0$ and $l>0$ has been observed before. It is referred to as the Nyquist limit. It is natural not to take more than this many random variables, or else one starts to repeat some of the previous random variables in various ways. Also, at $k=0$ or at $k=N/2$, the imaginary part becomes 0 and the real part gets a different variance to compensate, and we do not want to include these special cases.

We have now established that $\left(X_1,Y_1,X_2,Y_2,\ldots,X_m,Y_m\right)$ are iid $\mathcal{N}(0,N/2)$ random variables for $\mathcal{N}(0,N/2)$. In order to standardize the random variables, we define

$$\hat{X}_k=\sqrt{\frac{2}{N}}X_k \text{ and } \hat{Y}_k=\sqrt{\frac{2}{N}}Y_k \tag{16}$$

Then $\left(\hat{X}_1,\hat{Y}_1,\hat{X}_2,\hat{Y}_2,\ldots,\hat{X}_m,\hat{Y}_m\right)$ are iid $\mathcal{N}(0,1)$ random variables. Any subset of these random variables will also be a set of iid standard normal $\mathcal{N}(0,1)$ random variables. In particular, taking any $R<(N/2)$ and taking any set of size R as a subset of $\{1,\ldots,m\}$ we have $S=\{k_1,\ldots,k_R\}$

, for some cxoice of $k_1, \ldots, k_R$ where we may assume $1 < k_1 < k_2 < \ldots < k_R < N/2$ (eliminating unnecessary permutations). We define

$$Q_R = \sum_{r=1}^{R} \left(\hat{X}_{k_r}^2 + \hat{Y}_{k_r}^2 \right) \qquad (17)$$

Because the sum of k squared Normal random variables are distributed χ^2 with k degrees of freedom, the random variable Q_R is a Gamma $(2R, 2)$ - distributed random variable. We note that we can start from (17) and use (16) to rewrite

$$Q_R = \frac{2}{N} \sum_{r=1}^{R} \left(X_{k_r}^2 + Y_{k_r}^2 \right) \qquad (18)$$

Then $N/2 \cdot Q_R$ has a Gamma $(k = R, \theta = N)$ distribution. Another well established fact is that the square root of a Gamma-distributed random variable, say Q_R, has a Nakagami distribution if we set is shape parameter m of the Nakagami distribution equal to the shape parameter k of the Gamma distribution and the spread parameter Ω of the Nakagami distribution equal to the shape parameter times the spread parameter, $k \times \theta$, of a Gamma $k \times \theta$ distribution. Therefore, by defining

$$T_R = \sqrt{\sum_{r=1}^{R} \left(X_{k_r}^2 + Y_{k_r}^2 \right)},$$

We have that T_R is a Nakagami random variable with parameters $m = 1$ and $\Omega = NR$. Usually, one will choose to take just one harmonic, which is $R = 1$ but if one takes – for example - two or three harmonics ($R = 2$ or 3) then this theorem will provide a proper null hypothesis distribution assuming a white noise MRI signal to which we statistically compare our observed data.